SINGULARITIES OF THE STRESS-STRAIN STATE OF A PLATE IN THE NEIGHBORHOOD OF AN EDGE

(OSOBENNOSTI NAPRIAZHENNO-DEFORMIROVANNOGO SOSTOIANIIA PLITY V OKRESTNOSTI REBRA)

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In many problems in elasticity, the numerical evaluation of solutions requires that the behavior of the components in the stress-strain state be known in the neighborhood of singular points or lines on the surface of the body under consideration. This permits the approximation of the solution in the most convenient manner and the construction of an approximate process for its determination. The papers of Fufaev [1 and 2], and Kondrat'ev [3 and 4] are devoted to the solution of the Laplace, Poisson and elliptic equations in the regions having nonsmooth boundaries. Williams [5 and 6] and Ufliand [7] have established the character of stress singularities at the corner of a plane wedge for various boundary conditions on its edges. The aim of the present paper is to obtain the singularities of the state of stress in a nonhomogeneous plate in the neighborhood of edge points, i.e. points of intersection of the side surface with the face of a plate. The method used permits the determination of the character of the singularities without directly solving the boundary problem.

1. For greater generality, assume that the side surface Γ_2 is at an arbitrary angle α_2 $(0 < \alpha_2 \leq 2\pi)$ to the face Γ. The loading conditions on these surfaces in the neighborhood of the edge will be formulated below. In addition, let us assume that the plate is nonhomogeneous, and consists of two bodies which are rigidly joined along the cylindrical surface Γ_1, which passes through the plate edge L. The generator of this surface is inclined at an angle α_1 $(0 < \alpha_1 \leq 2\pi)$ to the plate surface (Fig. 1).

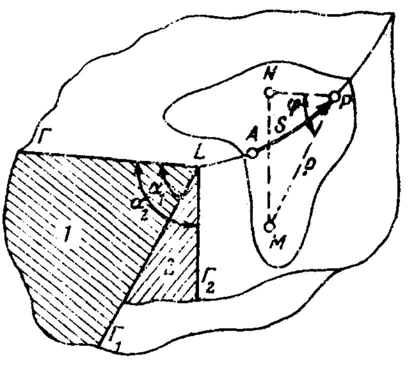

Fig. 1

Let G_1 and m_1 be, respectively, the shear modulus and Poisson's ratio for the material of the first body, bounded by the surfaces Γ and Γ_1, while G_2 and m_2 are the corresponding values for the second body, bounded by the surfaces Γ_1 and Γ_2.

Consider a sufficiently small neighborhood of point A on edge L. Introduce an orthogonal curvilinear coordinate system ρ, φ, s (Fig. 1).

Here MN is a perpendicular to the surface Γ from some point M lying inside the neighborhood under investigation: NP is the normal to the edge L, lying on that surface. The curvilinear coordinates of M are defined in the following manner: ρ is the distance from M to P; φ is the angle between NP and PM, and s is the distance from A to P measured along the curve L (the arrows on the sketch indicate the positive coordinate directions).

Let us write the equilibrium equations in this coordinate system

$$\frac{2(m-1)}{m-2}\left[-\frac{R(R-2\rho\cos\varphi)}{\rho^2(R-\rho\cos\varphi)^2}u_\rho+\frac{R-2\rho\cos\varphi}{\rho(R-\rho\cos\varphi)}\frac{\partial u_\rho}{\partial\rho}+\frac{\partial^2 u_\rho}{\partial\rho^2}\right]+$$
$$+\frac{3m-4}{m-2}\left[\frac{R\cos\varphi}{(R-\rho\cos\varphi)^2}\frac{\partial u_s}{\partial s}-\frac{1}{\rho^2}\frac{\partial u_\varphi}{\partial\varphi}\right]+\left[\frac{3m-4}{m-2}\rho\cos\varphi-R\right]\frac{\sin\varphi}{(R-\rho\cos\varphi)^2\rho}u_\varphi+$$
$$+\frac{m}{m-2}\left[\frac{R}{R-\rho\cos\varphi}\frac{\partial^2 u_s}{\partial\rho\,\partial s}+\frac{1}{\rho}\frac{\partial^2 u_\varphi}{\partial\rho\,\partial\varphi}+\frac{\sin\varphi}{R-\rho\cos\varphi}\frac{\partial u_\varphi}{\partial\rho}\right]-\frac{RR_s'\cos\varphi}{(R-\rho\cos\varphi)^3}u_s-$$
$$-\frac{R\rho\cos\varphi}{(R-\rho\cos\varphi)^3}\frac{\partial u_\rho}{\partial s}+\frac{R^2}{(R-\rho\cos\varphi)^2}\frac{\partial^2 u_\rho}{\partial s^2}+\frac{\sin\varphi}{\rho(R-\rho\cos\varphi)}\frac{\partial u_\rho}{\partial\varphi}+\frac{1}{\rho^2}\frac{\partial^2 u_\rho}{\partial\varphi^2}=0 \tag{1.1}$$

$$\frac{2(m-1)}{m-2}\left[\frac{R\sin\varphi}{\rho(R-\rho\cos\varphi)^2}u_\rho+\frac{\sin\varphi}{\rho(R-\rho\cos\varphi)}\frac{\partial u_\varphi}{\partial\varphi}+\frac{1}{\rho^2}\frac{\partial^2 u_\varphi}{\partial\varphi^2}\right]-$$
$$-\frac{3m-4}{m-2}\frac{R\sin\varphi}{(R-\rho\cos\varphi)^2}\frac{\partial u_s}{\partial s}+\frac{m}{m-2}\left[\frac{1}{\rho}\frac{\partial^2 u_\rho}{\partial\rho\,\partial\varphi}+\frac{R}{\rho(R-\rho\cos\varphi)}\frac{\partial^2 u_s}{\partial\varphi\,\partial s}\right]+$$
$$+\frac{1}{m-2}\frac{1}{\rho^2(R-\rho\cos\varphi)}\left\{\left[(3m-4)\frac{\rho R\cos\varphi}{R-\rho\cos\varphi}-(m-2)\frac{R^2}{R-\rho\cos\varphi}-\right.\right.$$
$$\left.\left.-2(m-1)\frac{\rho^2}{R-\rho\cos\varphi}\right]u_\varphi+[(3m-4)(R-\rho\cos\varphi)-m\rho\cos\varphi]\frac{\partial u_\rho}{\partial\varphi}\right\}+$$
$$+\frac{R-2\rho\cos\varphi}{\rho(R-\rho\cos\varphi)}\frac{\partial u_\varphi}{\partial\rho}+\frac{\partial^2 u_\varphi}{\partial\rho^2}-\frac{RR_s'\cos\varphi}{(R-\rho\cos\varphi)^3}\frac{\partial u_\varphi}{\partial s}+$$
$$+\frac{RR_s'\sin\varphi}{(R-\rho\cos\varphi)^3}u_s+\frac{R^2}{(R-\rho\cos\varphi)^2}\frac{\partial^2 u_\varphi}{\partial s^2}=0 \tag{1.2}$$

$$\frac{2(m-1)}{m-2}\frac{R}{R-\rho\cos\varphi}\left[-\frac{R_s'\rho\cos\varphi}{(R-\rho\cos\varphi)^2}\frac{\partial u_s}{\partial s}-\frac{R_s'\sin\varphi}{(R-\rho\cos\varphi)^2}u_\varphi+\frac{R}{R-\rho\cos\varphi}\frac{\partial^2 u_s}{\partial s^2}+\right.$$
$$\left.+\frac{R_s'\cos\varphi}{(R-\rho\cos\varphi)^2}u_\rho\right]+\frac{3m-4}{m-2}\frac{R\sin\varphi}{(R-\rho\cos\varphi)^2}\frac{\partial u_\varphi}{\partial s}+$$
$$+\frac{m}{m-2}\frac{R}{R-\rho\cos\varphi}\left[\frac{\partial^2 u_\rho}{\partial\rho\,\partial s}+\frac{1}{\rho}\frac{\partial^2 u_\varphi}{\partial s\,\partial\varphi}\right]-\frac{1}{(R-\rho\cos\varphi)^2}u_s+\frac{\sin\varphi}{\rho(R-\rho\cos\varphi)}\frac{\partial u_s}{\partial\varphi}+$$
$$+\frac{1}{\rho^2}\frac{\partial^2 u_s}{\partial\varphi^2}+\frac{R-2\rho\cos\varphi}{\rho(R-\rho\cos\varphi)}\frac{\partial u_s}{\partial\rho}+\frac{\partial^2 u_s}{\partial\rho^2}+$$
$$+\frac{1}{m-2}\frac{R}{R-\rho\cos\varphi}\left[-(3m-4)\frac{\cos\varphi}{R-\rho\cos\varphi}+\frac{m}{\rho}\right]\frac{\partial u_\rho}{\partial s}=0 \tag{1.3}$$

In Formulas (1.1) to (1.3), u_ρ, u_φ and u_s are the components of the displacement vector, taken in the directions of the introduced coordinates, while R is the radius of curvature L at point P.

Introduce a change of variables into (1.1) to (1.3)

$$\rho=e^{-t} \tag{1.4}$$

Equation (1.1) takes the form

$$\frac{2(m-1)}{m-2}\left[-\frac{R(R-2e^{-t}\cos\varphi)}{(R-e^{-t}\cos\varphi)^2}u_\rho-\frac{R-2e^{-t}\cos\varphi}{R-e^{-t}\cos\varphi}\frac{\partial u_\rho}{\partial t}+\frac{\partial^2 u_\rho}{\partial t^2}+\frac{\partial u_\rho}{\partial t}\right]+$$

$$+\frac{3m-4}{m-2}\left[\frac{R\cos\varphi e^{-2t}}{(R-e^{-t}\cos\varphi)^2}\frac{\partial u_s}{\partial s}-\frac{\partial u_\varphi}{\partial\varphi}\right]+\left[\frac{3m-4}{m-2}e^{-t}\cos\varphi-R\right]\frac{\sin\varphi e^{-t}}{(R-e^{-t}\cos\varphi)^2}u_\varphi-$$

$$-\frac{m}{m-2}\left[\frac{Re^{-t}}{R-e^{-t}\cos\varphi}\frac{\partial^2 u_s}{\partial t\,\partial s}+\frac{\partial^2 u_\varphi}{\partial t\,\partial\varphi}+\frac{e^{-t}\sin\varphi}{R-e^{-t}\cos\varphi}\frac{\partial u_\varphi}{\partial t}\right]-\frac{RR_s'e^{-2t}}{(R-e^{-t}\cos\varphi)^3}u_s-$$

$$-\frac{Re^{-3t}\cos\varphi}{(R-e^{-t}\cos\varphi)^3}\frac{\partial u_s}{\partial s}+\frac{R^2e^{-2t}}{(R-e^{-t}\cos\varphi)^2}\frac{\partial^2 u_\rho}{\partial s^2}+\frac{e^{-t}\sin\varphi}{R-e^{-t}\cos\varphi}\frac{\partial u_\rho}{\partial\varphi}+\frac{\partial^2 u_\rho}{\partial\varphi^2}=0$$

Noting that the neighborhood of A is sufficiently small so that terms containing the factor e^{-t} may be neglected in comparison with the rest, we obtain

$$\frac{2(m-1)}{m-2}\left[-u_\rho+\frac{\partial^2 u_\rho}{\partial t^2}\right]-\frac{3m-4}{m-2}\frac{\partial u_\varphi}{\partial\varphi}-\frac{m}{m-2}\frac{\partial^2 u_\varphi}{\partial t\,\partial\varphi}+\frac{\partial^2 u_\rho}{\partial\varphi^2}=0 \qquad (1.5)$$

Similar transformations of (1.2) and (1.3) yield

$$\frac{2(m-1)}{m-2}\frac{\partial^2 u_\varphi}{\partial\varphi^2}-\frac{m}{m-2}\frac{\partial^2 u_\rho}{\partial t\,\partial\varphi}-u_\varphi+\frac{3m-4}{m-2}\frac{\partial u_\rho}{\partial\varphi}+\frac{\partial^2 u_\varphi}{\partial t^2}=0 \qquad (1.6)$$

$$\frac{\partial^2 u_s}{\partial\varphi^2}+\frac{\partial^2 u_s}{\partial t^2}=0 \qquad (1.7)$$

Setting in (1.5) to (1.7) $m=m_i$, $u_\rho=u_{i\rho}$, $u_\varphi=u_{i\varphi}$ and $u_s=u_{is}$, we obtain, for $i=1,2$, a system of equilibrium equations for the first and second body, respectively. We seek solutions of the form

$$u_{i\rho}=e^{-tk}A_i(\varphi),\qquad u_{i\varphi}=e^{-tk}B_i(\varphi),\qquad u_{is}=e^{-tk_1}C_i(\varphi)\quad(i=1,2) \qquad (1.8)$$

The displacements are assumed to be bounded in the neighborhood of the edge, so that $k\geq 0$ and $k_1\geq 0$. Substituting (1.8) into equations of equilibrium, we obtain a system of differential equations for the determination of the functions $A_1(\varphi)$, $B_1(\varphi)$ and $C_1(\varphi)$

$$A_i''+\frac{m_ik-3m_i+4}{m_i-2}B_i'+\frac{2(m_i-1)}{m_i-2}(k^2-1)A_i=0 \qquad (1.9)$$

$$\frac{2(m_i-1)}{m_i-2}B_i''+\frac{m_ik+3m_i-4}{m_i-2}A_i'+(k^2-1)B_i=0\qquad(i=1,2) \qquad (1.10)$$

$$C_i''+k_1^2C_i=0 \qquad (1.11)$$

The general solutions of (1.9), (1.10) and (1.11) are easily found. For $k\neq 0$ and $k_1\neq 0$, they are given by the following relations:

$$A_i(\varphi)=C_{i1}(m_ik-3m_i+4)\cos(k-1)\varphi+$$

$$+C_{i2}(m_ik-3m_i+4)\sin(k-1)\varphi+C_{i3}\cos(k+1)\varphi+C_{i4}\sin(k+1)\varphi \qquad (1.12)$$

$$B_i(\varphi)=-C_{i1}(m_ik+3m_i-4)\sin(k-1)\varphi+$$

$$+C_{i2}(m_ik+3m_i-4)\cos(k-1)\varphi-C_{i3}\sin(k+1)\varphi+C_{i4}\cos(k+1)\varphi \qquad (1.13)$$

$$C_i(\varphi)=D_{i1}\sin k_1\varphi+D_{i2}\cos k_1\varphi\qquad(i=1,2) \qquad (1.14)$$

For $k=0$, the general solutions of (1.9) and (1.10) are represented in the form

$$A_i(\varphi)=(E_{i1}\varphi+E_{i2})\cos\varphi+(E_{i3}\varphi+E_{i4})\sin\varphi \qquad (1.15)$$

$$B_i(\varphi)=\left(E_{i3}\varphi+E_{i4}-\frac{m_i}{3m_i-4}E_{i1}\right)\cos\varphi-\left(E_{i1}\varphi+E_{i2}+\frac{m_i}{3m_i-4}E_{i3}\right)\sin\varphi\qquad(i=1,2) \qquad (1.16)$$

For $k_1 = 0$, the general solution of (1.11) is given by

$$C_i(\varphi) = F_{i1}\varphi + F_{i2} \qquad (i = 1,2) \tag{1.17}$$

The constants C_{1j}, D_{1j}, E_{1j} and F_{1j} are determined from the boundary conditions, which will now be formulated.

2. Let the surfaces Γ and Γ_2 in the neighborhood of point A under consideration be free of stresses. The governing equations on the contact surface Γ_1 are the equations for the components of a stress-strain state for two media. Since Γ, Γ_1 and Γ_2 are coordinate surfaces corresponding to $\varphi = 0$, $\varphi = \alpha_1$ and $\varphi = \alpha_2$, respectively, the boundary conditions are

$$\sigma_{1\varphi} = \tau_{1\rho\varphi} = \tau_{1s\varphi} = 0 \quad (\varphi = 0), \qquad \sigma_{2\varphi} = \tau_{2\rho\varphi} = \tau_{2s\varphi} = 0 \qquad (\varphi = \alpha_2) \tag{2.1}$$

$$\sigma_{1\varphi} = \sigma_{2\varphi}, \quad \tau_{1\rho\varphi} = \tau_{2\rho\varphi}, \quad \tau_{1s\varphi} = \tau_{2s\varphi}, \quad u_{1\rho} = u_{2\rho}, \quad u_{1\varphi} = u_{2\varphi}, \quad u_{1s} = u_{2s} \quad (\varphi = \alpha_1)$$

The indicated stresses, in terms of displacements in the above coordinate system, are given by

$$\sigma_{i\varphi} = \frac{2G_i}{m_i - 2}\left[\frac{m_i - 1}{\rho}\frac{\partial u_{i\varphi}}{\partial\varphi} + \left(m_i - \frac{R}{R - \rho\cos\varphi}\right)\frac{u_{i\rho}}{\rho} + \frac{\partial u_{i\rho}}{\partial\rho} + \right. \tag{2.2}$$

$$\left. + \frac{R}{R - \rho\cos\varphi}\frac{\partial u_{is}}{\partial s} + \frac{\sin\varphi}{R - \rho\cos\varphi}u_{i\varphi}\right]$$

$$\tau_{i\rho\varphi} = G_i\left[\frac{1}{\rho}\frac{\partial u_{i\rho}}{\partial\varphi} + \frac{\partial u_{i\varphi}}{\partial\rho} - \frac{u_{i\varphi}}{\rho}\right] \tag{2.3}$$

$$\tau_{is\varphi} = G_i\left[\frac{R}{R - \rho\cos\varphi}\frac{\partial u_{i\varphi}}{\partial s} + \frac{1}{\rho}\frac{\partial u_{is}}{\partial\varphi} - \frac{\sin\varphi}{R - \rho\cos\varphi}u_{is}\right] \qquad (i = 1, 2) \tag{2.4}$$

Introducing the changes of variable from (1.4) into (2.2) to (2.4) and taking into account the smallness of the neighborhood, we obtain the following relations:

$$\sigma_{i\varphi} = \frac{2G_i}{m_i - 2}e^t\left[(m_i - 1)\frac{\partial u_{i\varphi}}{\partial\varphi} + (m_i - 1)u_{i\rho} - \frac{\partial u_{i\rho}}{\partial t}\right] \tag{2.5}$$

$$\tau_{i\rho\varphi} = G_i e^t\left[\frac{\partial u_{i\rho}}{\partial\varphi} - \frac{\partial u_{i\varphi}}{\partial t} - u_{i\varphi}\right], \qquad \tau_{is\varphi} = G_i e^t\frac{\partial u_{is}}{\partial\varphi} \qquad (i = 1, 2) \tag{2.6}$$

Upon satisfying boundary conditions (2.1), we obtain a system of homogeneous equations in C_{1j} and D_{1j}

$$C_{11}m_1(k+1) + C_{13} = 0, \qquad C_{12}m_1(k-1) + C_{14} = 0 \tag{2.7}$$

$$C_{21}m_2(k+1)\cos(k-1)\alpha_2 + C_{22}m_2(k+1)\sin(k-1)\alpha_2 + C_{23}\cos(k+1)\alpha_2 + $$
$$+ C_{24}\sin(k+1)\alpha_2 = 0$$

$$-C_{21}m_2(k-1)\sin(k-1)\alpha_2 + C_{22}m_2(k-1)\cos(k-1)\alpha_2 - C_{23}\sin(k+1)\alpha_2 +$$
$$+ C_{24}\cos(k+1)\alpha_2 = 0$$

$$C_{11}(m_1k - 3m_1 + 4)\cos(k-1)\alpha_1 + C_{12}(m_1k - 3m_1 + 4)\sin(k-1)\alpha_1 +$$
$$+ C_{13}\cos(k+1)\alpha_1 + C_{14}\sin(k+1)\alpha_1 - C_{21}(m_2k - 3m_2 + 4)\cos(k-1)\alpha_1 -$$
$$-C_{22}(m_2k - 3m_2 + 4)\sin(k-1)\alpha_1 - C_{23}\cos(k+1)\alpha_1 - C_{24}\sin(k+1)\alpha_1 = 0$$

$$-C_{11}(m_1k + 3m_1 - 4)\sin(k-1)\alpha_1 + C_{12}(m_1k + 3m_1 - 4)\cos(k-1)\alpha_1 -$$
$$-C_{13}\sin(k+1)\alpha_1 + C_{14}\cos(k+1)\alpha_1 + C_{21}(m_2k + 3m_2 - 4)\sin(k-1)\alpha_1 -$$
$$-C_{22}(m_2k + 3m_2 - 4)\cos(k-1)\alpha_1 + C_{23}\sin(k+1)\alpha_1 - C_{24}\cos(k+1)\alpha_1 = 0$$

$$G_1[C_{11}m_1(k+1)\cos(k-1)\alpha_1 + C_{12}m_1(k+1)\sin(k-1)\alpha_1 + C_{13}\cos(k+1)\alpha_1 +$$
$$+ C_{14}\sin(k+1)\alpha_1] - G_2[C_{21}m_2(k+1)\cos(k-1)\alpha_1 + C_{22}m_2(k+1)\sin(k-1)\alpha_1 +$$
$$+ C_{23}\cos(k+1)\alpha_1 + C_{24}\sin(k+1)\alpha_1] = 0 \tag{2.8}$$

$$G_1 [-C_{11}m_1 (k-1) \sin (k-1)\alpha_1 + C_{12}m_1 (k-1) \cos (k-1)\alpha_1 - C_{13} \sin (k+1)\alpha_1 + C_{14} \cos (k+1)\alpha_1] - G_2 [-C_{21}m_2 (k-1) \sin (k-1)\alpha_1 + C_{22}m_2 (k-1) \cos (k-1)\alpha_1 - C_{23} \sin (k+1)\alpha_1 + C_{24} \cos (k+1)\alpha_1] = 0$$

$$D_{11} = 0, \qquad D_{21} \cos k_1\alpha_2 - D_{22} \sin k_1\alpha_2 = 0 \tag{2.9}$$

$$D_{11} \sin k_1\alpha_1 + D_{12} \cos k_1\alpha_1 - D_{21} \sin k_1\alpha_1 - D_{22} \cos k_1\alpha_1 = 0$$

$$G_1 (D_{11} \cos k_1\alpha_1 - D_{12} \sin k_1\alpha_1) - G_2 (D_{21} \cos k_1\alpha_1 - D_{22} \sin k_1\alpha_1) = 0$$

Setting the determinant of the system (2.8) and (2.9) equal to zero, we obtain, after some manipulation, the characteristic equations in k and k_1

$$G_1^2\left(\frac{m_2-1}{m_2}\right)^2[\sin^2 k\alpha_1 - k^2\sin^2\alpha_1] + G_2^2\left(\frac{m_1-1}{m_1}\right)^2[\sin^2 k(\alpha_2-\alpha_1) - k^2\sin^2(\alpha_2-\alpha_1)] +$$
$$+\left(\frac{G_2-G_1}{2}\right)^2[\sin^2 k\alpha_1 - k^2\sin^2\alpha_1][\sin^2 k(\alpha_2-\alpha_1) - k^2\sin^2(\alpha_2-\alpha_1)] +$$
$$+2G_1G_2\frac{m_1-1}{m_1}\frac{m_2-1}{m_2}[\sin k(\alpha_2-\alpha_1)\sin k\alpha_1\cos k\alpha_2 - k^2\sin\alpha_1\sin(\alpha_2-\alpha_1)\cos\alpha_2] +$$
$$+G_1(G_2-G_1)\frac{m_2-1}{m_2}[\sin^2 k\alpha_1 - k^2\sin^2\alpha_1]\sin^2 k(\alpha_2-\alpha_1) + \tag{2.10}$$
$$+G_2(G_1-G_2)\frac{m_1-1}{m_1}[\sin^2 k(\alpha_2-\alpha_1) - k^2\sin^2(\alpha_2-\alpha_1)]\sin^2 k\alpha_1 = 0 \qquad (k>0)$$

$$G_2 \cos k_1\alpha_1 \sin k_1(\alpha_2-\alpha_1) + G_1 \sin k_1\alpha_1 \cos k_1(\alpha_2-\alpha_1) = 0 \qquad (k_1>0) \tag{2.11}$$

Now consider the case of $k = 0$. Upon satisfying the boundary conditions (2.1), we obtain a system of equations in E_{ij} from which it follows that $E_{11} = E_{13} = 0$, $E_{22} = E_{12}$, $E_{24} = E_{14}$, with E_{12} and E_{14} being independent arbitrary constants. The components of the displacement vector are given by

$$u_{2\rho} = u_{1\rho} = E_{12}\cos\varphi + E_{14}\sin\varphi, \qquad u_{2\varphi} = u_{1\varphi} = E_{14}\cos\varphi - E_{12}\sin\varphi \tag{2.12}$$

It is easily seen that the expressions in (2.12) represent rigid body displacements.

Similarly, for $k_1 = 0$, we have $F_{11} = 0$, $F_{22} = F_{12}$, and the solution $u_{1s} = u_{2s} = F_{12}$ also represents a rigid body displacement.

Thus, the cases $k = 0$ and $k_1 = 0$ are of no interest in the problem at hand. We will now investigate the solutions corresponding to positive values of k and k_1, defined by the realtions (2.10) and (2.11).

In the most prevalent case of contact between two bodies, for $\alpha_1 = \frac{1}{2}\pi$ and $\alpha_2 = \frac{3}{2}\pi$, investigation of the singularity of the solution in the neighborhood of edge AB (Fig. 2) yields the characteristic equations in k and k_1.

$$G_1^2\left(\frac{m_2-1}{m_2}\right)^2\left(\sin^2\frac{k\pi}{2} - k^2\right) + G_2^2\left(\frac{m_1-1}{m_1}\right)^2\sin^2 k\pi +$$
$$+\left(\frac{G_2-G_1}{2}\right)^2\left(\sin^2\frac{k\pi}{2} - k^2\right)\sin^2 k\pi + 2G_1G_2\frac{m_1-1}{m_1}\frac{m_2-1}{m_2}\times$$
$$\times\sin k\pi\sin\frac{k\pi}{2}\cos\frac{3k\pi}{2} + G_1(G_2-G_1)\frac{m_2-1}{m_2}\left(\sin^2\frac{k\pi}{2} - k^2\right)\sin^2 k\pi +$$
$$+G_2(G_1-G_2)\frac{m_1-1}{m_1}\sin^2 k\pi\sin^2\frac{k\pi}{2} = 0 \tag{2.13}$$

$$G_2 \cos\frac{k_1\pi}{2}\sin k_1\pi + G_1 \sin\frac{k_1\pi}{2}\cos k_1\pi = 0 \tag{2.14}$$

Let us now examine the case of a homogeneous plate the side surface of which makes an angle α with the plane of the face. This case may be obtained by setting $G_2 = G_1 = G$, $m_2 = m_1 = m$ and $\alpha_2 = \alpha$ in the relations previously derived. The characteristic equations take the form

$$\sin^2 k\alpha - k^2 \sin^2\alpha = 0 \qquad (k > 0) \tag{2.15}$$

$$\sin k_1\alpha = 0 \qquad (k_1 > 0) \tag{2.16}$$

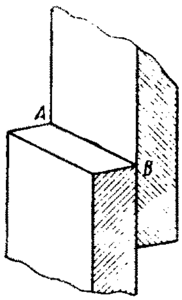

Fig. 2

Whereupon, Equations (2.8) and (2.9) yield

$$C_{1j} = C_{2j}, \qquad D_{1j} = D_{2j}$$

Setting

$$C_{1j} = C_{2j} = C_j, \quad D_{1j} = D_{2j} = D_j$$

we obtain a system of equations for the constants C_j and D_j

$$C_3 = -C_1 m (k+1), \qquad C_4 = -C_2 m (k-1) \tag{2.17}$$

$$C_1 (k+1) \sin k\alpha \sin\alpha + C_2 (\sin k\alpha \cos\alpha - k \sin\alpha \cos k\alpha) = 0$$

$$C_1 (k \sin\alpha \cos k\alpha + \sin k\alpha \cos\alpha) + C_2 (k-1) \sin k\alpha \sin\alpha = 0$$

$$D_1 = 0, \qquad D_2 \sin k_1\alpha = 0 \tag{2.18}$$

It is easily found by taking note of (2.15) that, for all $\alpha \neq \pi$ in the interval $(0, 2\pi)$, the rank of the matrix of system (2.17) is three, while for $\alpha = \pi$ and $\alpha = 2\pi$ the matrix is of rank 2. Hence, the case $\alpha = \pi$ and $\alpha = 2\pi$ will be examined separately. For $0 < \alpha < \pi$ and $\pi < \alpha < 2\pi$, we have

$$C_2 (k-1) = -C_1 (k \cot k\alpha + \cot\alpha), \quad C_3 = -C_1 m (k+1), \quad C_4 = C_1 m (k \cot k\alpha + \cot\alpha) \tag{2.19}$$

Then we obtain for the components of the displacement vector, when $0 < \alpha < \pi$ and $\pi < \alpha < 2\pi$

$$u_\rho = C\rho^k [(mk - 3m + 4)(k-1)\cos(k-1)\varphi - (mk - 3m + 4)(k \cot k\alpha + \cot\alpha)\sin(k-1)\varphi - m(k^2-1)\cos(k+1)\varphi + m(k-1)(k \cot k\alpha + \cot\alpha)\sin(k+1)\varphi] \tag{2.20}$$

$$u_\varphi = C\rho^k [-(mk + 3m - 4)(k-1)\sin(k-1)\varphi - (mk + 3m - 4)(k \cot k\alpha + \cot\alpha)\cos(k-1)\varphi + m(k^2-1)\sin(k+1)\varphi + m(k-1)(k \cot k\alpha + \cot\alpha)\cos(k+1)\varphi] \tag{2.21}$$

$$u_z = D_2 \rho^{k_1} \cos k_1\varphi \tag{2.22}$$

Here k and k_1 are determined from relations (2.15) and (2.16), respectively.

If $\alpha = \pi$, the characteristic equations for k and k_1 are

$$\sin k\pi = 0, \qquad \sin k_1\pi = 0 \tag{2.23}$$

i.e. k and k_1 are positive integers. Clearly, in that case the stresses in the neighborhood of the edge are finite, as expected.

For $\alpha = 2\pi$, the characteristic equation for k and k_1 are

$$\sin 2k\pi = 0, \qquad \sin 2k_1\pi = 0 \tag{2.24}$$

Evidently, in this case, as the plate edge is approached, the stresses increase without bounds, except for $k = \frac{1}{2}$ or $k_1 = \frac{1}{2}$. The displacement vector components are here given by

$$u_\rho = {}^1/_2 \sqrt{\rho}\, \{-C_1[(5m - 8)\cos {}^1/_2\varphi + 3m \cos {}^3/_2 \xi\varphi] + C_2 [(5m - 8)\sin {}^1/_2\varphi + m \sin {}^3/_2 \varphi]\}$$

$$u_{\varphi} = {}^1/_2 \sqrt{\rho} \{C_1[(7m - 8) \sin {}^1/_2 \varphi + 3 m \sin {}^3/_2 \varphi] + $$
$$+ C_2 [(7m - 8)\cos {}^1/_2 \varphi + m \cos {}^3/_2 \varphi]\}$$
$$u_s = D_2 \sqrt{\rho} \cos {}^1/_2\varphi \qquad (2.25)$$

Thus, when the surfaces Γ and Γ_2 are unloaded in the neighborhood of the edge, the singularities are of the form ρ^{k-1} or ρ^{k_1-1}, where k and k_1 are obtained from (2.10) and (2.11).

3. Now let the surfaces Γ and Γ_2 be rigidly clamped in the neighborhood under consideration, i.e.

$$u_{1\rho} = u_{1\varphi} = u_{1s} = 0 \quad (\varphi = 0), \qquad u_{2\rho} = u_{2\varphi} = u_{2s} = 0 \quad (\varphi = \alpha_2) \qquad (3.1)$$

$$\sigma_{1\varphi} = \sigma_{2\varphi}, \quad \tau_{1\rho\varphi} = \tau_{2\rho\varphi}, \quad \tau_{1s\varphi} = \tau_{2s\varphi}, \quad u_{1\rho} = u_{2\rho}, \quad u_{1\varphi} = u_{2\varphi}, \quad u_{1s} = u_{2s} \qquad (\varphi = \alpha_1)$$

Satisfying these conditions for $k > 0$ and $k_1 > 0$ in a manner similar to the above, we obtain a system of homogeneous equations in C_{1j} and D_{1j}. Setting the determinants of these systems equal to zero, we obtain the characteristic equations for k and k_1

$$G_2^2 \left(\frac{m_2 - 1}{3m_2 - 4}\right)^2 \left[\sin^2 k\alpha_1 - \left(\frac{m_1 k}{3m_1 - 4}\right)^2 \sin^2 \alpha_1\right] +$$
$$+ G_1^2 \left(\frac{m_1 - 1}{3m_1 - 4}\right)^2 \left[\sin^2 k(\alpha_2 - \alpha_1) - \left(\frac{m_2 k}{3m_2 - 4}\right)^2 \sin^2(\alpha_2 - \alpha_1)\right] +$$
$$+ \left(\frac{G_2 - G_1}{2}\right)^2 \left[\sin^2 k\alpha_1 - \left(\frac{m_1 k}{3m_1 - 4}\right)^2 \sin^2 \alpha_1\right]\left[\sin^2 k(\alpha_2 - \alpha_1) - \left(\frac{m_2 k}{3m_2 - 4}\right)^2 \times\right.$$
$$\left.\times \sin^2(\alpha_2 - \alpha_1)\right] - G_2(G_2 - G_1)\frac{m_2 - 1}{3m_2 - 4}\left[\sin^2 k\alpha_1 - \left(\frac{m_1 k}{3m_1 - 4}\right)^2 \sin^2 \alpha_1\right] \sin^2 k(\alpha_2 - \alpha_1) -$$
$$- G_1(G_1 - G_2)\frac{m_1 - 1}{3m_1 - 4}\left[\sin^2 k(\alpha_2 - \alpha_1) - \left(\frac{m_2 k}{3m_2 - 4}\right)^2 \sin^2(\alpha_2 - \alpha_1)\right] \sin^2 k\alpha_1 + \qquad (3.2)$$
$$+ 2G_1 G_2 \frac{m_1 - 1}{3m_1 - 4}\frac{m_2 - 1}{3m_2 - 4}\left[\sin k\alpha_1 \sin k(\alpha_2 - \alpha_1) \cos k\alpha_2 -\right.$$
$$\left.- \frac{m_1}{3m_1 - 4}\frac{m_2}{3m_2 - 4} k^2 \sin \alpha_1 \sin(\alpha_2 - \alpha_1) \cos \alpha_2\right] = 0 \qquad (k > 0)$$

$$G_1 \cos k_1\alpha_1 \sin k_1(\alpha_2 - \alpha_1) + G_2 \sin k_1\alpha_1 \cos k_1(\alpha_2 - \alpha_1) = 0 \qquad (k_1 > 0) \qquad (3.3)$$

If either of the quantities k or k_1 is taken equal to zero, then the corresponding displacements become zero, as expected. Hence, hereinafter we will assume k and k_1 to be positive.

For $\alpha_1 = {}^1/_2 \pi$, $\alpha_2 = {}^3/_2 \pi$ (Fig. 2), the characteristic equations are

$$G_2^2 \left(\frac{m_2 - 1}{3m_2 - 4}\right)^2 \left[\sin^2 \frac{k\pi}{2} - \left(\frac{m_1 k}{3m_1 - 4}\right)^2\right] + G_1^2 \left(\frac{m_1 - 1}{3m_1 - 4}\right)^2 \sin^2 k\pi +$$
$$+ \left(\frac{G_2 - G_1}{2}\right)^2 \left[\sin^2 \frac{k\pi}{2} - \left(\frac{m_1 k}{3m_1 - 4}\right)^2\right] \sin^2 k\pi - G_2(G_2 - G_1)\frac{m_2 - 1}{3m_2 - 4}\left[\sin^2 \frac{k\pi}{2} -\right.$$
$$\left.- \left(\frac{m_1 k}{3m_1 - 4}\right)^2\right] \sin^2 k\pi - G_1(G_1 - G_2)\frac{m_1 - 1}{3m_1 - 4} \sin^2 k\pi \sin^2 \frac{k\pi}{2} +$$
$$+ 2G_1 G_2 \frac{m_1 - 1}{3m_1 - 4}\frac{m_2 - 1}{3m_2 - 4} \sin \frac{k\pi}{2} \sin k\pi \cos \frac{3k\pi}{2} = 0 \qquad (k > 0) \qquad (3.4)$$

$$G_1 \cos \frac{k_1\pi}{2} \sin k_1\pi + G_2 \sin \frac{k_1\pi}{2} \cos k_1\pi = 0 \qquad (k_1 > 0) \qquad (3.5)$$

If the plate material is homogeneous, we have the following equations in k and k_1 :

$$\sin^2 k\alpha - \left(\frac{mk}{3m-4}\right)^2 \sin^2\alpha = 0 \qquad (k > 0) \tag{3.6}$$

$$\sin k_1\alpha = 0 \qquad (k_1 > 0) \tag{3.7}$$

The equations for C_j and D_j, for $0 < \alpha < \pi$ and $\pi < \alpha < 2\pi$, become

$$C_3 = -(mk - 3m + 4)\, C_1, \qquad C_4 = -(mk + 3m - 4)\, C_2 \tag{3.8}$$

$$C_1 (mk - 3m + 4) - C_2\, [mk \cot k\alpha + (3m - 4) \cot\alpha] = 0 \tag{3.9}$$

$$C_1 (mk \cot k\alpha - (3m - 4) \cot\alpha] + C_2\, (mk + 3m - 4) = 0 \tag{3.10}$$

$$D_2 = 0, \qquad D_1 \sin k_1\alpha = 0 \tag{3.11}$$

It is not difficult to show that if

$$\alpha = \frac{m}{m-2}\,\frac{n\pi}{2} \qquad (n = 1, 2, 3) \tag{3.12}$$

then both coefficients in Equation (3.10) vanish for $k = -(3m-4)/m$, which turns out to be a root of Equation (3.6) in this case. The coefficients in (3.9) are nonzero for $k = -(3m-4)/m$, but if

$$\alpha = \frac{m}{2(m-1)}\,\frac{n\pi}{2} \qquad (n = 1, 2, 3, \ldots, 7), \tag{3.13}$$

both become zero for $k = (3m-4)/m$.

Hence, when α satisfies condition (3.12), the following formulas must be used for determining u_ρ and u_φ, when $k = -(3m-4)/m$:

$$u_\rho = -C\rho^{-\frac{3m-4}{m}} \left[\left(\cot\alpha + \cot\frac{3m-4}{m}\alpha\right)\cos\frac{4(m-1)}{m}\varphi + 2\sin\frac{4(m-1)}{m}\varphi - \right.$$
$$\left. - \left(\cot\alpha + \cot\frac{3m-4}{m}\alpha\right)\cos\frac{2(m-2)}{m}\varphi\right] \tag{3.14}$$

$$u_\varphi = C\rho^{-\frac{3m-4}{m}} \left(\cot\alpha + \cot\frac{3m-4}{m}\alpha\right)\sin\frac{2(m-2)}{m}\varphi$$

For all values of k different from $-(3m-4)/m$ but satisfying (3.12) as well as for arbitrary k not satisfying (3.12), the displacements u_ρ and u_φ are given by

$$u_\rho = C\rho^k\{(mk - 3m + 4)(mk + 3m - 4)\cos(k-1)\varphi - (mk - 3m + 4)\,[mk \cot k\alpha -$$
$$- (3m - 4)\cot\alpha]\sin(k-1)\varphi - (mk - 3m + 4)(mk + 3m - 4)\cos(k+1)\varphi +$$
$$+ (mk + 3m - 4)\,[mk \cot k\alpha - (3m - 4)\cot\alpha]\sin(k+1)\varphi\} \tag{3.15}$$

$$u_\varphi = C\rho^k\{-(mk + 3m - 4)^2 \sin(k-1)\varphi - (mk + 3m - 4)\,[mk \cot k\alpha -$$
$$- (3m - 4)\cot\alpha]\cos(k-1)\varphi + (mk - 3m + 4)(mk + 3m - 4)\sin(k+1)\varphi +$$
$$+ (mk + 3m - 4)[mk \cot k\alpha - (3m - 4)\cot\alpha]\cos(k+1)\varphi\}$$

The displacement u_s for arbitrary $0 < \alpha < \pi$ and $\pi < \alpha < 2\pi$ is given by

$$u_s = D_1\rho^{k_1}\sin k_1\varphi \tag{3.16}$$

If $\alpha = \pi$, the stresses in the neighborhood of the edge are finite for this problem also, as one might expect.

For $\alpha = 2\pi$, the characteristic equations for k and k_1 are

$$\sin 2k\pi = 0 \qquad (k > 0), \qquad \sin 2k_1\pi = 0 \qquad (k_1 > 0) \tag{3.17}$$

As in the preceding case, the associated stresses increase without bounds as the edge of the plate is approached, except for $k = \frac{1}{2}$ or $k_1 = \frac{1}{2}$. The expressions for the displacements in this case are

$$u_\rho = {}^1/_2 \sqrt{\rho}\{-C_1(5m-8)(\cos {}^1/_2\varphi - \cos {}^3/_2\varphi) + C_2[(5m-8)\sin {}^1/_2\varphi - (7m-8)\sin {}^3/_2\varphi]\} \qquad (3.18)$$

$$u_\varphi = {}^1/_2 \sqrt{\rho}\{C_1[(7m-8)\sin {}^1/_2\varphi - (5m-8)\sin {}^3/_2\varphi] + C_2(7m-8)(\cos {}^1/_2\varphi - \cos {}^3/_2\varphi)\}, \; u_s = D_2\sqrt{\rho}\sin {}^1/_2\varphi$$

Thus, in the case of rigidly clamped surfaces Γ and Γ_2, the stresses in the neighborhood of the edge have singularities of the form ρ^{k-1} and ρ^{k_1-1}, where k and k_1 are given by relations (3.2) and (3.3), respectively.

4. We now consider two more sets of boundary conditions. In the first case, the surface Γ is free while the surface Γ_2 is rigidly clamped. In the second case, the surface Γ_2 is free while the surface Γ is clamped. Proceeding as before, we obtain, in the first case, the characteristic equations for positive k and k_1

$$-G_2\left(\frac{m_2-1}{m_2}\right)^2\frac{3m_1-4}{m_1}\left[\sin^2 k\alpha_1 - \frac{4(m_1-1)^2 - m_1^2k^2\sin^2\alpha_1}{m_1(3m_1-4)}\right] - \qquad (4.1)$$
$$- G_2\left(\frac{m_1-1}{m_1}\right)^2\frac{3m_2-4}{m_2}\left[\sin^2 k(\alpha_2-\alpha_1) - \frac{4(m_2-1)^2 - m_2^2k^2\sin^2(\alpha_2-\alpha_1)}{m_2(3m_2-4)}\right] +$$
$$+ (G_1-G_2)\frac{m_2-1}{m_2}\frac{3m_2-4}{m_2}[\sin^2 k\alpha_1 - k^2\sin^2\alpha_1]\sin^2 k(\alpha_2-\alpha_1) +$$
$$+ (G_1-G_2)\frac{m_1-1}{m_1}\left(\frac{3m_2-4}{m_2}\right)^2\left[\sin^2 k(\alpha_2-\alpha_1) - \left(\frac{m_2k}{3m_2-4}\right)^2\sin^2(\alpha_2-\alpha_1)\right]\sin^2 k\alpha_1 +$$
$$+ \frac{(G_1-G_2)^2}{4G_2}\left(\frac{3m_2-4}{m_2}\right)^2[\sin^2 k\alpha_1 - k^2\sin^2\alpha_1]\left[\sin^2 k(\alpha_2-\alpha_1) - \left(\frac{m_2k}{3m_2-4}\right)^2\sin^2(\alpha_2-\alpha_1)\right] -$$
$$- 4G_2\left(\frac{m_1-1}{m_1}\right)^2\left(\frac{m_2-1}{m_2}\right)^2 - 2G_1\frac{m_1-1}{m_1}\frac{m_2-1}{m_2}\Big\{k^2\sin\alpha_1\sin(\alpha_2-\alpha_1)\cos\alpha_2 -$$
$$- \frac{3m_2-4}{m_2}\sin k\alpha_1\sin k(\alpha_2-\alpha_1)\left[2\frac{G_2}{G_1}\sin k\alpha_1\sin k(\alpha_2-\alpha_1) - \cos k(2\alpha_1-\alpha_2)\right]\Big\} = 0$$

$$G_2\cos k_1\alpha_1\cos k_1(\alpha_2-\alpha_1) - G_1\sin k_1\alpha_1\sin k_1(\alpha_2-\alpha_1) = 0 \qquad (4.2)$$

The characteristic equations for the second case may be obtained from Equations (4.1) and (4.2) by interchanging α_1, m_1, G_1 with $\alpha_2-\alpha_1$, m_2, G_2, respectively. For $k = 0$ or $k_1 = 0$, the corresponding displacements again vanish.

For the case of contact between two bodies as shown in Fig. 2, the characteristic equations for k and k_1 may be obtained in a similar manner from Equations (4.1) and (4.2) by setting $\alpha_1 = \frac{1}{2}\pi$ and $\alpha_2 = \frac{3}{2}\pi$. In investigating a homogeneous plate, the equations for k and k_1 are for both cases

$$\sin^2 k\alpha - \frac{4(m-1)^2 - m^2k^2\sin^2\alpha}{m(3m-4)} = 0, \qquad \cos k_1\alpha = 0 \qquad (4.3)$$

The components of the displacement vector for both the first and second case of a clamped surface are

$$u_\rho = C\rho^k[L_k(\alpha)(mk-3m+4)\cos(k-1)\varphi + M_k(\alpha)(mk-3m+4)\sin(k-1)\varphi - L_k(\alpha)\,m(k+1)\cos(k+1)\varphi - M_k(\alpha)m(k-1)\sin(k+1)\varphi]$$

$$u_\varphi = C\rho^k[-L_k(\alpha)(mk+3m-4)\sin(k-1)\varphi + M_k(\alpha)(mk+3m-4)\cos(k-1)\varphi + L_k(\alpha)\,m(k+1)\sin(k+1)\varphi - M_k(\alpha)\,m(k-1)\cos(k+1)\varphi]$$

$$u_s = D_2\rho^{k_1}\cos k_1\varphi \qquad (4.4)$$

$$u_\rho = C\rho^k[P_k(\alpha)(mk-3m+4)\cos(k-1)\varphi + R_k(\alpha)(mk-3m+4)\sin(k-1)\varphi - P_k(\alpha)(mk-3m+4)\cos(k+1)\varphi - R_k(\alpha)(mk+3m-4)\sin(k+1)\varphi]$$

$$u_\varphi = C\rho^k[-P_k(\alpha)(mk+3m-4)\sin(k-1)\varphi + R_k(\alpha)(mk+3m-4)\cos(k-1)\varphi +$$
$$+ P_k(\alpha)(mk-3m+4)\sin(k+1)\varphi - R_k(\alpha)(mk+3m-4)\cos(k+1)\varphi]$$

Here
$$u_s = D_1\rho^{k_1}\sin k_1\varphi \qquad (4.5)$$

$$L_k(\alpha) = (mk-2m+2)\sin\alpha\cos k\alpha + (m-2)\cos\alpha\sin k\alpha$$
$$M_k(\alpha) = (mk-m+2)\sin\alpha\sin k\alpha - 2(m-1)\cos\alpha\cos k\alpha \qquad (4.6)$$
$$P_k(\alpha) = (mk+2m-2)\sin\alpha\cos k\alpha + (m-2)\cos\alpha\sin k\alpha$$
$$R_k(\alpha) = (mk-m+2)\sin\alpha\sin k\alpha + 2(m-1)\cos\alpha\cos k\alpha$$

Thus, for the cases of mixed boundary conditions examined above, the stresses in the neighborhood of the edge have singularities of the form ρ^{k-1} or ρ^{k_1-1}, where k and k_1 are obtained from equations of the form (4.1) and (4.2).

In conclusion, let us note that the characteristic equations (2.15), (3.6) and (4.3) for a homogeneous plate coincide with the equations obtained by Ufliand [7] in investigating the corresponding problems for a plane wedge. This is only natural, for clearly the method at hand divides the procedure for finding the solution to the posed three-dimensional problem into solving one separate problem for the displacement vector component u_s and another problem for the components u_ρ and u_φ, the latter being the same as for a plane wedge. The singularity for the torsion problem could not, of course, be developed in [7].

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